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On Oscillation of Second-Order Retarded Equations

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1. INTRODUCTION

The oscillatory behavior of the homogeneous part of the equation

$$(r(t)\,y'(t))' + a(t)\,y(t-\tau(t)) = f(t) \tag{1}$$

has been studied by numerous authors [2, 6, 5, 7] under the restrictive assumption that $a(t)$ be eventually negative on some half-line $[T, \infty)$, $T > 0$. Nothing much seems to be known for the case when $a(t)$ is allowed to alternate sign for arbitrarily large values of t. Recently, Bhatia [1] considered the ordinary differential equation

$$(r(t)\,y'(t))' + a(t)\,y(t) = 0 \tag{2}$$

and proved an oscillation theorem for Eq. (2) under the relaxed condition that $a(t)$ be oscillatory on $[0, \infty)$. Bhatia's technique does not extend to cover Eq. (1).

The purpose in this paper is to present a theorem that gives the oscillatory property of the bounded solutions of Eq. (1) under one of the assumptions that $a(t)$ be oscillatory on some half-line $[T, \infty)$.

We call a function on $[T, \infty)$ oscillatory, if it has arbitrarily large zeros. Otherwise we call it nonoscillatory.

In what follows, only nontrivial continuous solutions of Eq. (1) extendable on some half-line $[T_1, \infty)$, $T_1 \geqslant T$ will be considered throughout this paper. In addition the following assumptions will be made for the rest of this paper.

Assumptions

(i) $\tau(t)$ and $r(t)$ are nonnegative and continuous functions on the whole real line R and $f(t)$ is continuous on R

$$r(t) \geqslant p > 0, \qquad \int^{\infty} |f(t)|\, dt < \infty, \qquad 0 \leqslant \tau(t) \leqslant M$$

for some positive constant M and $t \in R$.

(ii) $a: R \to R$ and continuous. $a^{+}(t) = \max(a(t), 0)$.

2. Main Results

THEOREM 1. *Suppose*

$$\int^{\infty} \frac{1}{r(t)}\, dt < \infty \qquad \text{and} \qquad \int^{\infty} a^{+}(t)\, dt = \infty, \tag{3}$$

$$\int^{\infty} a^{-}(t)\, dt < \infty. \tag{4}$$

Let $y(t)$ be a bounded solution of Eq. (1). *Then either $y(t)$ is oscillatory or $y(t)$ tends to a finite limit as $t \to \infty$.*

Proof. Suppose $y(t)$ is nonoscillatory. Then $y(t)$ eventually attains a constant sign. Without any loss suppose there exists a T_1 such that for $t > T_1$, $y(t) > 0$. The case when $y(t) < 0$ can be handled similarly. Let $T_2 = T_1 + M$, so that $y(t - \tau(t)) > 0$ for $t \geqslant T_2$. Integrating (1) between T_2 and t we have

$$r(t)\, y'(t) - r(T_2)\, y'(T_2) + \int_{T_2}^{t} a^{+}(s)\, y(s - \tau(s))\, ds - \int_{T_2}^{t} a^{-}(s)\, y(s - \tau(s))\, ds = \int_{T_2}^{t} f(s)\, ds. \tag{5}$$

Case 1.

$$\lim_{t\to\infty} \int_{T_2}^{t} a^{+}(s)\, y(s - \tau(s))\, ds = \infty.$$

If so, Eq. (5) reveals, $r(t)\, y'(t) \to -\infty$ as $t \to \infty$ since

$$\int_{T_2}^{\infty} |f(t)|\, dt < \infty \qquad \text{and} \qquad \int_{T_2}^{\infty} a^{-}(s)\, y(s - \tau(s))\, ds < \infty$$

due to boundedness of $y(t)$ and condition (4). Since $r(t) > 0$ for all t, we must have $y'(t) < 0$ for $t \geqslant T_2$ and $y(t)$ decreases to a finite limit.

Case 2.

$$\lim_{t\to\infty} \int_{T_2}^{\infty} a^+(t)\, y(t-\tau(t))\, dt < \infty.$$

If so, then

$$\lim_{t\to\infty} \int_{T_2}^{\infty} a^+(t) = \infty$$

implies

$$\liminf_{t\to\infty} y(t) = 0. \tag{6}$$

If $\lim_{t\to\infty} y(t)$ does not exist then let

$$\limsup_{t\to\infty} y(t) = k > 0. \tag{7}$$

Due to (6) and (7) we can assume that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with the following properties

(i) $t_n \to \infty$ as $n \to \infty$, $t_n > T_2$ for all n. $\int_{t_1}^{\infty} |f(t)|\, dt < \epsilon$, $\epsilon > 0$ is arbitrary.

(ii) $y(t_n) \to 0$ as $n \to \infty$.

(iii) There exist points $s_n \in [t_{n-1}, t_n]$ such that $\lim_{n\to\infty} y(s_n) = k$.

$$y(s_n) = k_n = \max y(t) \qquad \text{for } t \in [t_{n-1}, t_n].$$

Let

$$z_n = \inf\{s_n : y(s_n) = k_n,\, s_n \in [t_{n-1}, t_n]\}.$$

By continuity of $y(t)$ and definition of infema of a set, it follows that

$$y(z_n) = k_n. \tag{8}$$

(iv) Let a_n be the largest point before z_n and b_n be the smallest point after z_n such that

$$y(a_n) = y(b_n) = k_n/2. \tag{9}$$

Then by definition of a_n and b_n

$$y(t) > k_n/2 \qquad \text{for } t \in (a_n, b_n). \tag{10}$$

We will show that

$$\liminf_{n\to\infty} (b_n - a_n) > 0.$$

In order to do that let $\epsilon > 0$ be arbitrarily small and let $N > T_2$ be a large enough zero of $y'(t)$ so that

$$\int_N^\infty a^-(t)\, y(t-\tau(t))\, dt \leqslant \epsilon/2 \tag{11}$$

and

$$\int_N^\infty |f(t)|\, dt \leqslant \epsilon/2. \tag{12}$$

Integrating Eq. (1) on $[N, t]$ we have

$$r(t)\, y'(t) + \int_N^t a^+(s)\, y(s-\tau(s))\, ds = \int_N^t f(s)\, ds + \int_N^t a^-(s)\, y(s-\tau(s))\, ds, \tag{13}$$

$$r(t)\, y'(t) < \epsilon/2 + \epsilon/2 = \epsilon. \tag{14}$$

Now let $S > N > t$ be another zero of $y'(t)$. Integrating Eq. (1) again on $[t, S]$ we have

$$-r(t)\, y'(t) + \int_t^S a^+(s)\, y(s-\tau(s))\, ds = \int_t^S f(s)\, ds + \int_t^S a^-(s)\, y(s-\tau(s))\, ds \tag{15}$$

or

$$r(t)\, y'(t) \geqslant -\epsilon. \tag{16}$$

Now (14) and (16) imply

$$\lim_{t\to\infty} r(t)\, y'(t) = 0 \tag{17}$$

and since $r(t) \geqslant p > 0$ for all t, we have

$$\lim_{t\to\infty} y'(t) = 0. \tag{18}$$

We now consider the interval $[a_n, z_n]$. By the mean-value theorem,

$$y'(\xi_n) = \frac{y(z_n) - y(a_n)}{z_n - a_n} = \frac{k_n - k_n/2}{z_n - a_n} \geqslant \frac{k_n/2}{b_n - a_n}, \qquad a_n < \xi_n < z_n. \tag{19}$$

Now if

$$\liminf_{n\to\infty}(b_n - a_n) = 0,$$

then (19) will violate (18). This is the required contradiction. Thus

$$\liminf_{n\to\infty}(b_n - a_n) > 0. \tag{20}$$

Due to the choice of a_n and b_n in $[t_{n-1}, t_n]$, it follows that

$$y'(a_n) \geqslant 0 \qquad \text{and} \qquad y'(b_n) \leqslant 0. \tag{21}$$

In fact if $y'(a_n) < 0$, then continuity of $y'(t)$ implies that $y(t)$ decreases to the right of a_n and since $y(z_n) = k_n$, there will be a point $t_{n_0} > a_n$, $z_n > t_{n_0}$ such that $y(t_{n_0}) = k_n/2$ which is a contradiction to the choice of a_n. Hence $y'(a_n) \geqslant 0$ and similarly $y'(b_n) \leqslant 0$. Now we shall use a generalized version of a technique given in [3] to complete the proof.

$$\frac{k_n}{2} = \int_{a_n}^{z_n} y'(t)\,dt, \tag{22}$$

$$-\frac{k_n}{2} = \int_{z_n}^{b_n} y'(t)\,dt. \tag{23}$$

From (22) and (23) we get

$$k_n \leqslant \int_{a_n}^{z_n} |y'(t)|\,dt + \int_{z_n}^{b_n} |y'(t)|\,dt = \int_{a_n}^{b_n} |y'(t)|\,dt. \tag{24}$$

This yields

$$\begin{aligned}(k_n)^2 &\leqslant \left(\int_{a_n}^{b_n} |y'(t)|\,dt\right)^2 \\ &= \left[\int_{a_n}^{b_n} \frac{1}{\sqrt{r}} \sqrt{r}\,|y'|\,dt\right]^2 \\ &\leqslant \int_{a_n}^{b_n} \frac{1}{r(t)}\,dt \int_{a_r}^{b_n} (r(t)\,y'(t))\,y'(t)\,dt\end{aligned}$$

by Schwarz's inequality. Thus integrating by parts we get

$$\begin{aligned}k_n{}^2 &\leqslant \int_{a_n}^{b_n} \frac{1}{r(t)}\,dt \left[r(b_n)\,y'(b_n)\,y(b_n) - r(a_n)\,y'(a_n)\,y(a_n) - \int_{a_n}^{b_n} (ry')'\,y(t)\,dt\right] \\ &= \int_{a_n}^{b_n} \frac{1}{r(t)}\,dt \left[\left\{k_{n/2}(r(b_n)\,y'(b_n) - r(a_n)\,y'(a_n)) - \int_{a_n}^{b_n} y(t)\,f(t)\,dt\right\}\right. \\ &\quad \left. + \int_{a_n}^{b_n} a(t)\,y(t-\tau(t))\,y(t)\,dt\right]\end{aligned} \tag{25}$$

by Eq. (1). Since

$$y'(b_n) \leqslant 0, \qquad y'(a_n) \geqslant 0, \qquad r(t) > 0$$

and

$$\int_{a_n}^{b_n} y(t) \, |f(t)| \, dt > 0,$$

it follows from (25)

$$k_n^2 \leqslant \int_{a_n}^{b_n} \frac{1}{r(t)} \left[\int_{a_n}^{b_n} a(t) \, y(t - \tau(t)) \, y(t) \, dt + \int_{a_n}^{b_n} y(t) \, |f(t)| \, dt \right]$$

or

$$k_n^2 \leqslant \int_{a_n}^{b_n} \frac{1}{r(t)} \, dt \left[\int_{a_n}^{b_n} a^+(t) \, y(t - \tau(t)) \, k_n \, dt + k_n \int_{a_n}^{b_n} |f(t)| \, dt \right] \tag{26}$$

from which we get

$$\left(k_n \Big/ \int_{a_n}^{b_n} \frac{1}{r(t)} \, dt \right) \leqslant \int_{a_n}^{b_n} a^+(t) \, y(t - \tau(t)) \, dt + \int_{a_n}^{b_n} |f(t)| \, dt. \tag{27}$$

Now

$$\begin{aligned} \infty &> \int_{t_1}^{\infty} a^+(t) \, y(t - \tau(t)) \, dt \\ &> \int_{a_1}^{\infty} a^+(t) \, y(t - \tau(t)) \, dt \\ &\geqslant \sum_{n=1}^{\infty} \int_{a_n}^{b_n} a^+(t) \, y(t - \tau(t)) \, dt. \end{aligned}$$

Making use of (27) and

$$\int_{t_1}^{\infty} |f(t)| \, dt < \epsilon,$$

we get

$$\infty > \int_{t_1}^{\infty} a^+(t) \, y(t - \tau(t)) \, dt \geqslant \sum_{n=1}^{\infty} \left(k_n \Big/ \int_{a_n}^{b_n} \frac{1}{r} \, dt \right) - \epsilon. \tag{28}$$

Since

$$\lim_{n \to \infty} \int_{a_n}^{b_n} \frac{1}{r(t)} \, dt = 0,$$

it follows from (28) $k_n \to 0$ as $n \to \infty$. This is a contradiction since $\lim_{n \to \infty} k_n = k$.

The proof is now complete.

THEOREM 2. *Suppose $a(t) \geqslant 0$, then every nonoscillatory solution of Eq.* (1) *tends to a finite limit if $\int^{\infty} a(t)\,dt = \infty$ and $\int^{\infty} (1/r)\,dt < \infty$.*

Proof. The only place we made use of the boundedness of $y(t)$ was in Case 1 of the proof of Theorem 1. Since $a^{-}(t) \equiv 0$, the rest of the proof of Theorem 2 holds.

In Case 2 of that proof, we actually proved that $y(t) \to 0$ as $t \to \infty$. The proof of Theorem 2 is now complete.

Remark 1. Our Theorem 2 is similar to the theorem of [4, p. 93] and in a way somewhat better. The fact that we don't require $a(t) \geqslant k > 0$ seems to be more practical. However, Theorem 2 doesn't generalize the theorem of [4].

EXAMPLE. Consider the equation

$$(t^2 y'(t))' + [1/(t-1)]\, y(t-1) = 1/(t-1)^2, \qquad t \geqslant 2$$

then $1/t$ is a nonoscillatory solution that tends to zero.

$$r(t) \equiv t^2, \qquad f(t) = 1/(t-1)^2, \qquad a(t) = 1/(t-1).$$

The conditions of Theorem 2 are satisfied.

COROLLARY 1. *Every bounded solution of*

$$(ry')' + a(t)\, y(t - \tau(t)) = 0 \tag{29}$$

either tends to a finite limit or oscillates.

Proof. Follows from Theorem 1.

3. MORE RESULTS

The proof of Theorem 1 leads to a stronger result for the equation

$$y''(t) + a(t)\, y(t) = f(t). \tag{30}$$

In fact we will prove the following theorem.

THEOREM 3. *Suppose*

$$\int^{\infty} a^{-}(t)\,dt < \infty, \qquad \lim_{n\to\infty} \int_{c_n}^{d_n} a^{+}(t)\,dt = \infty$$

where $c_n > d_n$ are two sequences such that $c_n \to \infty$, $d_n \to \infty$ and $(d_n - c_n) \to \infty$ as $n \to \infty$. Then every bounded solution of Eq. (30) *is either oscillatory or tends to zero.*

Proof. We proceed as in Theorem 1 and find out that Case 1 of the proof of Theorem 1 drops out. In Case 2, inequality (27) becomes

$$\frac{k_n}{b_n - a_n} \leqslant \int_{a_n}^{b_n} a^+(t)\, y(t)\, dt + \int_{a_n}^{b_n} |f(t)|\, dt. \tag{31}$$

Now

$$\limsup_{n\to\infty}(b_n - a_n) < \infty.$$

Because if

$$\limsup_{n\to\infty}(b_n - a_n) = \infty,$$

then from Case 2 of the proof of Theorem 1 we have

$$\begin{aligned} \infty &> \int_{t_1}^{\infty} a^+(t)\, y(t)\, dt \\ &> \limsup_{n\to\infty} \int_{a_n}^{b_n} a^+(t)\, y(t)\, dt \\ &\geqslant \limsup_{n\to\infty} \frac{k_n}{2} \int_{a_n}^{b_n} a^+(t) = \infty. \end{aligned}$$

This is a contradiction unless $k_n \to 0$ as $n \to \infty$. Hence

$$\limsup_{n\to\infty}(b_n - a_n) < \infty.$$

The rest of the proof now follows from inequality (28).

THEOREM 4. *If* $a(t) \geqslant 0$ *and satisfies the conditions of Theorem* 3, *then every nonoscillatory solution of Eq.* (30) *tends to zero as* $t \to \infty$.

Proof. Follows from Theorem 3.

Remark. For the case $r(t) \equiv 1$, Theorem 4 is an improvement of theorem of Hammett [4].

The following example is not covered by Hammett's theorem but our Theorem 3 apply to it. Consider

$$y''(t) + \frac{\sin^2 t}{2 + \sin t}\, y(t) = e^{-t}(\sin^2 t - 2\cos t + 2), \qquad t \geqslant 0$$

has $y = (2 + \sin t)\, e^{-t}$ as the nonoscillatory solution that tends to zero as $t \to \infty$. Conditions of Theorem 3 are satisfied.

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